Anyons and the Landau problem in the noncommutative plane

Mikhail S. Plyushchay*

Departamento de Física, Universidad de Santiago de Chile
Casilla 307, Santiago 2, Chile

Abstract

The Landau problem in the noncommutative plane is discussed in the context of realizations of the two-fold centrally extended planar Galilei group and the anyon theory.

In 2+1 dimensions, Galilei group admits a two-fold central extension [1, 2] characterized by the algebra with the nonzero Poisson bracket relations

$$\{\mathcal{K}_i, \mathcal{P}_j\} = m\delta_{ij}, \qquad \{\mathcal{K}_i, \mathcal{K}_j\} = -\kappa\epsilon_{ij}, \tag{1}$$

$$\{\mathcal{K}_i, \mathcal{H}\} = \mathcal{P}_i, \qquad \{\mathcal{J}, \mathcal{P}_i\} = \epsilon_{ij}\mathcal{P}_j, \qquad \{\mathcal{J}, \mathcal{K}_i\} = \epsilon_{ij}\mathcal{K}_j, \tag{2}$$

where m and κ are the central charges. The algebra has the two Casimir elements

$$\mathcal{C}_1 = m\mathcal{J} + \kappa\mathcal{H} - \epsilon_{ij}\mathcal{K}_i\mathcal{P}_j, \qquad \mathcal{C}_2 = m\mathcal{H} - \frac{1}{2}\mathcal{P}_i^2, \tag{3}$$

which correspond to the (multiplied by the mass m) internal angular momentum (spin) and energy.

There are two possibilities to realize this algebra as a symmetry of a free particle on a plane: the *minimal* realization and the *extended* one [cf. the two formulations for a free relativistic anyon [3]]. Requiring that the particle coordinate X_i forms a Galilei covariant object with respect to the action of the generators J, $\mathcal{P}_i$ and $\mathcal{K}_i$, treating the Galilei generators as integrals of motion and identifying the $\mathcal{P}_i$ as the canonical momentum p_i, and, finally, putting the spin and internal energy to be equal to zero ($\mathcal{C}_1 = \mathcal{C}_2 = 0$), we arrive at the following realization of the generators:

$$\mathcal{P}_i = p_i, \qquad K_i = mX_i - tp_i + m\theta\epsilon_{ij}p_j, \qquad \mathcal{J} = \epsilon_{ij}X_ip_j + \frac{1}{2}\theta\vec{p}^{\,2}, \qquad \mathcal{H} = \frac{1}{2m}\vec{p}^{\,2}, \tag{4}$$

$\theta = \kappa/m^2$. As a result, the X_i has a usual free particle evolution, $\dot{X}_i = \frac{1}{m}p_i$. The price we pay for such a minimal realization of the exotic Galilei algebra is the non-commutativity of the coordinate components

$$\{X_i, X_j\} = \theta\epsilon_{ij}, \tag{5}$$

and the non-canonical form of the associated symplectic structure

$$\sigma_0 = dp_i \wedge dX_i + \frac{1}{2}\theta\epsilon_{ij}dp_i \wedge dp_j. \tag{6}$$

One can define another sort of the coordinate [4, 5],

$$Y_i = X_i + \theta\epsilon_{ij}p_j. \tag{7}$$

It has the same bracket with p_j,

$$\{Y_i, p_j\} = \delta_{ij}, \tag{8}$$

*E-mail: mplyushc@lauca.usach.cl

and, hence, the same evolution law as the coordinate X_i. In terms of the Y_i and X_i, the symplectic structure and angular momentum are diagonal,

$$\sigma_0 = \frac{1}{2\theta}\epsilon_{ij}\left(dY_i \wedge dY_j - dX_i \wedge dX_j\right), \qquad \mathcal{J} = \frac{1}{2\theta}\left(Y_i^2 - X_i^2\right).$$

On the other hand, in terms of the Y_i and p_i the boost generator is represented in the usual form $\mathcal{K}_i = mY_i - tp_i$. However, the Y_i, unlike the X_i, is not covariant with respect to the Galilei boosts, $\{\mathcal{K}_i, Y_j\} = t\delta_{ij} - m\theta\epsilon_{ij}$. As we shall see below, the importance of the coordinate (7) reveals under coupling the system to the external electric and magnetic fields.

Due to the noncommutative nature of the both X_i and Y_i, there is no coordinate representation associated with them. But since

$$\{Y_i, Y_j\} = -\theta\epsilon_{ij}, \qquad \{X_i, Y_j\} = 0, \tag{9}$$

one can define the third sort of the coordinate,

$$\mathcal{X}_i = \frac{1}{2}(X_i + Y_i). \tag{10}$$

It has commuting components and reduces the symplectic structure and angular momentum to a canonical form,

$$\sigma_0 = dp_i \wedge d\mathcal{X}_i, \qquad \mathcal{J} = \epsilon_{ij}\mathcal{X}_i p_j.$$

Like the Y_i, the coordinate $\mathcal{X}_i$ is not covariant with respect to the Galilean boosts, $\{\mathcal{K}_i, \mathcal{X}_j\} = t\delta_{ij} - \frac{1}{2}m\theta\epsilon_{ij}$. The importance of this third coordinate is that at the quantum level it provides us with the Schrödinger representation, $\hat{\mathcal{X}}_i\Psi(\mathcal{X}) = \mathcal{X}_i\Psi(\mathcal{X})$, $\hat{p}_i = -i\partial_i\Psi(\mathcal{X})$. In this representation in accordance with Eqs. (10), (7) the action of the covariant coordinate operator is reduced to the star multiplication [6]:

$$\hat{X}_i\Psi(\mathcal{X}) = \left(\mathcal{X}_i - \frac{i}{2}\theta\epsilon_{ij}\partial_j\right)\Psi(\mathcal{X}) \equiv \mathcal{X}_i \star \Psi(\mathcal{X}).$$

We conclude that in the minimal realization of the exotic Galilei group the coordinate of the free particle cannot be commutative and covariant simultaneously, cf. the case of the anyons [3]. There exist at least three sorts of the coordinate, each of which has definite advantages and disadvantages.

Duval and Horvathy showed [2] that within the minimal realization, the coupling of the particle to the arbitrary external electric and magnetic fields can be achieved via a simple generalization of the free symplectic structure and Hamiltonian for

$$\sigma_{em} = dp_i \wedge dX_i + \frac{1}{2}\theta\epsilon_{ij}dp_i \wedge dp_j + \frac{1}{2}eB(X)\epsilon_{ij}dX_i \wedge dX_j, \qquad H_{em} = \frac{1}{2m}\vec{p}^{\,2} + eV(X), \tag{11}$$

where $V(X)$ is a scalar potential associated with the electric field $E_i = -\partial_i V(X)$. The Poisson brackets corresponding to the σ_{em} are

$$\{X_i, X_j\} = \frac{\theta}{1 - e\theta B}\epsilon_{ij}, \qquad \{X_i, p_j\} = \frac{1}{1 - e\theta B}\delta_{ij}, \qquad \{p_i, p_j\} = \frac{eB}{1 - e\theta B}\epsilon_{ij}, \tag{12}$$

and the equations of motion for X_i and p_i take the form similar to the $\theta = 0$ case but with the mass m changed for the effective mass $m^* = m(1 - e\theta B)$. The essential property of the coordinate Y_i defined by Eq. (7) is that it has the same brackets (8), (9) in the presence of any magnetic field $B(X)$ [4].

It is obvious that in the case of the critical value of the magnetic field $B = B_c \equiv (e\theta)^{-1}$, for which symplectic form (11) degenerates while brackets (12) blow up and the effective mass m^* disappears, has to be treated separately [2, 4]. In [4] it was shown that in this case the system realizes a Hall-like motion, which is described by the coordinate Y_i. On the other hand, it is clear that in a generic case of the inhomogeneous magnetic field there is a problem with realization of the operators satisfying the quantum analogs of the Poisson bracket relations (12).

The simultaneous commutativity and covariance of the coordinate can be incorporated into the theory via the extended realization of the exotic Galilei group [7, 4]. This is achieved by supplying the phase space with the two additional canonically conjugate translation-invariant variables v_i associated with an infinite-component Majorana-type representation of the exotic planar Galilei group, being analogous to the Dirac α matrices. The symplectic structure is given here by

$$\sigma = dp_i \wedge dx_i + \frac{1}{2}\kappa\epsilon_{ij}dv_i \wedge dv_j, \tag{13}$$

and the rotation and the boost generators are realized in the form

$$\mathcal{J} = \epsilon_{ij}x_ip_j + \frac{1}{2}\kappa v_i^2, \qquad \mathcal{K}_i = mx_i - tp_i + \kappa\epsilon_{ij}v_j, \tag{14}$$

while as before, the translation generator is identified with p_i. Require that the first Casimir element from (3) takes zero value. Then, with taking into account (14), we fix the form of the Hamiltonian,

$$\mathcal{H} = \vec{p}\vec{v} - \frac{1}{2}m\vec{v}^{\,2}, \tag{15}$$

and find the equations of motion generated by it,

$$\dot{x}_i = v_i, \qquad \dot{p}_i = 0, \qquad \dot{v}_i = \omega\epsilon_{ij}(v_j - m^{-1}p_j), \tag{16}$$

where $\omega = m/\kappa$. Like in the case of the Dirac equation, Hamiltonian (15) is linear in momenta, the velocities are noncommuting, $\{v_i, v_j\} = -\kappa^{-1}\epsilon_{ij}$, and in the evolution of the covariant coordinate x_i, $\{x_i, x_j\} = 0$, there appears a Zitterbewegung-like term:

$$x_i(t) = X_i(0) + \frac{1}{m}p_it - \omega^{-1}\epsilon_{ij}V_j(t),$$

where

$$X_i = x_i + \frac{\kappa}{m}\epsilon_{ij}V_j, \tag{17}$$

$$V_i = v_i - m^{-1}p_i, \tag{18}$$

and $V_i(t) = (\cos\omega t \cdot \delta_{ij} + \sin\omega t \cdot \epsilon_{ij})V_j(0)$. The quantities V_i form a planar vector invariant with respect to the space translations and boosts, $\{\mathcal{K}_i, V_j\} = 0$, $\{p_i, V_j\} = 0$, and can be associated with the internal rotation.

The quantity (17) has the same transformation properties under the action of $\mathcal{P}_i$, $\mathcal{K}_i$ and $\mathcal{J}$ as the coordinate x_i. Unlike the x_i, it is Zitterbewegung-free, $\dot{X}_i = m^{-1}p_i$, and has the noncommuting components, $\{X_i, X_j\} = \theta\epsilon_{ij}$ [cf. the properties of the covariant coordinate X_i within the minimal realization]. The X_i is analogous to the Foldy-Wouthuysen coordinate for the Dirac particle. The combination $\mathcal{X}_i = X_i - \frac{1}{2}\theta\epsilon_{ij}p_j$ (with X_i given by (17)) is also Zitterbewegung-free, it has commuting components, but is not covariant under the action of the Galilei boosts [cf. the properties of the coordinate (10)]. It is analogous to the Newton-Wigner coordinate for the Dirac particle [8].

It is interesting to note that the dynamical picture of the extended formulation turns out to be exactly the same as that for the usual planar particle ($\theta = 0$) subjected to the external homogeneous magnetic and electric fields [5].

The Hamiltonian and the rotation generator are represented equivalently in the form

$$H = \frac{1}{2m}\vec{p}^{\,2} - \frac{1}{2}m\vec{V}^2, \tag{19}$$

$$\mathcal{J} = \epsilon_{ij}X_ip_j + \frac{1}{2}\theta\vec{p}^{\,2} + \frac{1}{2}\kappa\vec{V}^2,$$

while the boost generator takes the same form as in (4) with X_i given by Eq. (17). We have not fixed yet the second Casimir element, which is reduced here to the integral of motion associated with the

Zitterbewegung (circular motion), $\mathcal{C}_2 = m^2\vec{V}^2$. Such a Hamiltonian system corresponds to a special non-relativistic limit applied to the model of relativistic particle with torsion [9] associated with the (2+1)-dimensional analog of the Majorana equation and underlying the theory of relativistic anyons [8]. Like the relativistic analog, the present system is described by the higher-derivative Lagrangian

$$L = \frac{1}{2}m\dot{x}_i^2 + \theta\epsilon_{ij}\dot{x}_i\ddot{x}_j, \tag{20}$$

which was analysed by Lukierski, Stichel and Zakrzewski [10] (ignoring its relation to the relativistic higher-derivative model [9]). In accordance with the Ostrogradski theory of higher-derivative systems, at the Hamiltonian level the velocity components $\dot{x}_i$ are identified as independent phase space variables v_i.

From the structure of the Hamiltonian (19) and equivalent form of the symplectic structure (13),

$$\sigma = dp_i \wedge dX_i + \frac{1}{2}\theta\epsilon_{ij}dp_i \wedge dp_j + \frac{\kappa}{2}\epsilon_{ij}dV_i \wedge dV_j, \tag{21}$$

it is clear that the system (20) describes not a free particle in the noncommutative plane but a sort of rotator with degrees of freedom of the ghost nature since they contribute a negative kinetic term into the Hamiltonian. In order to reduce this system to a free exotic particle of Duval and Horvathy [2] (which corresponds to a minimal realization of the two-fold centrally extended Galilei group), it is sufficient to fix the second Casimir element by introducing the second class constraints $V_i = 0$, $i = 1, 2$ [4]. From the point of view of such a reduction, the coordinate (17) is the extension of the initial coordinate x_i commuting with the second class constraints [5].

There is also another possibility to reduce the system (20), preserving the linear in the momentum Hamiltonian structure (15) similar to that of the Dirac equation. Instead of the two second class constraints, the physical subspace of the system can be singled out by imposing a complex polarization condition given by one first class complex constraint

$$V_- = 0, \tag{22}$$

$V_- = V_1 - iV_2$. Then at the quantum level a state of the system can be decomposed into the series in the Fock space states associated with the velocity variables $\hat{v}_\pm = \hat{v}_1 \pm i\hat{v}_2$, $|\Psi\rangle = \sum_{k=0}^{\infty}\psi_k|k\rangle_v$, where $\hat{v}_-|0\rangle_v = 0$. As a result, the quantum system will be described by the pair of the infinite-component wave equations [7]

$$i\partial_t\psi_k + \sqrt{\frac{k+1}{2\theta}}\,\frac{\hat{p}_+}{m}\,\psi_{k+1} = 0, \tag{23}$$

$$\hat{p}_-\psi_k + \sqrt{\frac{2(k+1)}{\theta}}\,\psi_{k+1} = 0, \tag{24}$$

where $k = 0, 1, \ldots,$ and $\hat{p}_\pm = \hat{p}_1 \pm i\hat{p}_2$. Eq. (23) is the Schrödinger equation corresponding to the classical Hamiltonian (15), while Eq. (24) is the quantum analog of the classical constraint (22), whose role is to separate effectively only one independent physical field degree of freedom. The set (23), (24) has the sense of the infinite-component wave equations of the Dirac-Majorana-Levy-Leblond type for the exotic particle, associated with the two-fold central extension of the planar Galilei group. It was obtained in [7] by applying a special Jackiw-Nair non-relativistic limit [11] to the spinor set of the equations proposed earlier in [12] for the description of relativistic anyons.

Having in mind the discussed nature of the coordinates which appear in the minimal realization of the exotic Galilei group, it is clear that the coupling prescription (11) in the case of the Dirac theory would correspond to the minimal coupling in terms of the Foldy-Wouhtuysen coordinates. Since the extended formulation of a free exotic particle results in the free wave equations (23), (24) realized in terms of the commuting covariant coordinates x_i, it is natural to expect that the coupling of the system to external electric and magnetic fields proceeding from the extended formulation would be more close in nature to the usual minimal coupling prescription of the Dirac theory.

The coupling of the exotic particle to external electric and magnetic fields in the extended formulation can be realized as follows [4]. Modify the complex polarization condition (22) via

the minimal coupling prescription, $p_i \to P_i = p_i - eA_i(x)$, $\epsilon_{ij}\partial_i A_j = B$. Then the generalization of the Hamiltonian (15) can be fixed from the requirement of its (weak) commutativity with the changed polarization condition. The essential feature of such a coupling scheme is that the two real constraints

$$\Lambda_i = v_i - \frac{1}{m}P_i = 0, \qquad \{\Lambda_i, \Lambda_j\} = -\kappa^{-1}(1-\beta)\epsilon_{ij}, \tag{25}$$

$\beta = \beta(x) = e\theta B(x)$, corresponding to one complex polarization condition, change their nature from the second class into the first class constraints at the critical value of the magnetic field, $B = B_c$. As a result, at $B = B_c$, the constraints (25) eliminate not one but two degrees of freedom, leaving only one degree described effectively by the noncommutative coordinate Y_i [4]. In a generic case, the classical Hamiltonian weakly commuting with constraints (25) and reducing to the Hamiltonian (15) in the free case, has the form $\tilde{\mathcal{H}} = H_B + U$, with

$$H_B = \frac{1}{1-\beta}(P_i - \beta v_i)v_i - \frac{1}{2}mv_i^2, \tag{26}$$

and U being an arbitrary function of X_i, or Y_i.

In the case of homogeneous magnetic field different from the critical one and for zero electric field ($U = 0$), the obtained system describes the Landau problem in the noncommutative plane. It is necessary to distinguish the cases of subcritical and overcritical magnetic fields. Assume that $e\theta > 0$. Then the physical states for $B < B_c$ are separated by the quantum polarization condition

$$\hat{\Lambda}_-|\Psi\rangle = 0. \tag{27}$$

The solutions of Eq. (27) describe the physical states of the form

$$|\Psi\rangle_{phys} = \exp\left(\frac{1}{2}\theta m\hat{P}_-\hat{v}_+\right)(|0\rangle_v|\psi\rangle), \tag{28}$$

where $|0\rangle_v$, $\hat{v}_-|0\rangle_v = 0$, is the vacuum state of the Fock space generated by the velocity operators, and $|\psi\rangle$ is a velocity-independent state associated with other degrees of freedom. The action of the Hamiltonian operator corresponding to (26) is reduced on the states (28) to

$$\hat{H}_B|\Psi\rangle_{phys} = \exp\left(\frac{1}{2}\theta m\hat{P}_-\hat{v}_+\right)\left(|0\rangle_v\hat{H}_*|\psi\rangle\right), \qquad \hat{H}_* = \frac{1}{2m^*}\hat{P}_+\hat{P}_-. \tag{29}$$

For $B < 0$, the spectrum of the system is characterized by the energy values $E_N = e|B|N/m^*$, $N = 0, 1, \ldots$, and by the angular momentum values $j = N, N-1, \ldots$. For $0 < B < B_c$, $E_N = e|B|(N+1)/m^*$, $N = 0, 1, \ldots$, and $j = -N, -N+1, \ldots$ [4]. The structure of the physical states is essentially different for $B < 0$ and $0 < B < B_c$: in the former case, the finite number of the velocity Fock space states $|n\rangle_v$, $n = 0, \ldots, N$, contribute to a physical state, while in the latter case all the infinite tower of the velocity Fock states ($n = 0, 1, \ldots$) contributes to it. It is essential, however, that in the both cases the common eigenstates of the energy and angular momentum are normalisable. In the critical case, due to the first class nature of the constraints (25), equation(27) should be supplemented with the quantum condition $\hat{\Lambda}_+|\Psi\rangle = 0$. The solutions of these two equations are given by the wave functions proposed by Laughlin to describe the ground states in the fractional quantum Hall effect [13], and coincide with the solutions of the equation (27) taken in the limit $B \to B_c$, for the details see ref. [4].

In the case of overcritical magnetic field $B > B_c$ the solutions of the quantum equation (27) are not normalisable [4]. The reason of this is rooted in a simple observation. In accordance with Eq. (25), the brackets between constraints Λ_i, $i = 1, 2$, for $B > B_c$ have an opposite sign in comparison with the subcritical case $B < B_c$. It means that the operator $\hat{\Lambda}_-$ being an annihilation-like operator for $B < B_c$, transforms into the creation-like operator having no nontrivial kernel for $B > B_c$. Therefore, in the overcritical case, the physical states have to be separated by the quantum condition $\hat{\Lambda}_+|\Psi\rangle = 0$ instead of the condition (27). This change has to be accompanied by the change of the direction of time, $t \to -t$ [4, 14].

It was observed in [4] that in a generic case of inhomogeneous magnetic field the quantum analog of the classical Hamiltonian (26) commuting with the quantum condition (27) has a nonlocal nature. On the other hand, one notes that there exists a class of the quantum systems with coordinate-dependent mass related to some quasi-exactly solvable systems [15]. This, probably, indicates that for inhomogeneous magnetic field of a special form the problem of non-locality of the quantum Hamiltonian can be solved using some ideas related to quasi-exact solvability and supersymmetry [16].

Since the exotic particle system in the noncommutative plane is related via a special non-relativistic limit to the relativistic anyon, this means that the phenomenon similar to the existence of the critical magnetic field should also exist if one couples the latter system to the external electromagnetic field. The problem of non-locality should also reveal itself there for electromagnetic field of a generic form.

Acknowledgements. The work was supported by the FONDECYT-Chile (project 1050001). I thank the organizers of the SQS-05 Workshop (JINR, Dubna, Russia) and the Summer Mini-Workshop in Theoretical Physics (2006) (CECS, Valdivia, Chile) for hospitality.

References

[1] J.-M. Lévy-Leblond, *Galilei group and Galilean invariance.* In: *Group Theory and Applications* (Loebl Ed.), **II**, Acad. Press, New York, p. 222 (1972); A. Ballesteros, M. Gadella and M. del Olmo, *Journ. Math. Phys.* **33** (1992) 3379; Y. Brihaye, C. Gonera, S. Giller and P. Kosiński, `hep-th/9503046`.

[2] C. Duval and P. A. Horváthy, *Phys. Lett.* **B 479** (2000) 284 [`hep-th/0002233`]; *Journ. Phys.* **A 34** (2001) 10097 [`hep-th/0106089`]

[3] J. L. Cortes and M. S. Plyushchay, *Int. J. Mod. Phys.* **A 11** (1996) 3331 [`hep-th/9505117`].

[4] P. A. Horvathy and M. S. Plyushchay, *Nucl. Phys.* **B 714** (2005) 269 [`hep-th/0502040`].

[5] M. A. del Olmo and M. S. Plyushchay, *Ann. Phys. (NY)*, in press [`hep-th/0508020`].

[6] D. Bigatti and L. Susskind, *Phys. Rev.* **D 62** (2000) 066004 [`hep-th/9908056`]; L. Mezincescu, `hep-th/0007046`.

[7] P. A. Horvathy and M. S. Plyushchay, *Phys. Lett.* **B 595** (2004) 547 [`hep-th/0404137`].

[8] P. A. Horváthy and M. S. Plyushchay, *JHEP* **0206** (2002) 033 [`hep-th/0201228`].

[9] M. S. Plyushchay, *Phys. Lett.* **B262** (1991) 71; *Nucl. Phys.* **B 362** (1991) 54.

[10] J. Lukierski, P. C. Stichel, W. J. Zakrzewski, *Annals of Physics (N. Y.)* **260** (1997) 224 [`hep-th/9612017`].

[11] R. Jackiw and V. P. Nair, *Phys. Lett.* **B 480** (2000) 237 [`hep-th/0003130`].

[12] M. S. Plyushchay, *Phys. Lett.* **B 320** (1994) 91 [`hep-th/9309148`]; *Annals Phys. (NY)* **245** (1996) 339 [`hep-th/9601116`]; *Mod. Phys. Lett.* **A 12** (1997) 1153 [`hep-th/9705034`].

[13] R. B. Laughlin, *Phys. Rev. Lett.* **50** (1983) 1395; S. Girvin and T. Jach, *Phys. Rev.* **B29** (1984) 5617.

[14] P. A. Horvathy, `hep-th/0602133`.

[15] B. Bagchi, P. Gorain, C. Quesne, R. Roychoudhury, *Europhys. Lett.* 72 (2005) 155 [`quant-ph/0505171`].

[16] S. M. Klishevich and M. S. Plyushchay, *Nucl. Phys.* **B 606** (2001) 583 [`hep-th/0012023`].